Theoretical Computer Science 412 (2011) 4636–4649

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Disjoint path covers in recursive circulants $G(2^m, 4)$ with faulty elements[☆]

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ARTICLE INFO

Article history:
Received 21 June 2010
Received in revised form 1 March 2011
Accepted 26 April 2011
Communicated by G. Ausiello

Keywords:
Disjoint paths
Fault-hamiltonicity
Graph theory
Embedding
Linear arrays
Fault tolerance
Interconnection networks

ABSTRACT

A *k-disjoint path cover* of a graph is defined as a set of k internally vertex-disjoint paths connecting given sources and sinks in such a way that every vertex of the graph is covered by a path in the set. In this paper, we analyze the k-disjoint path cover of recursive circulant $G(2^m, 4)$ under the condition that at most f faulty vertices and/or edges are removed. It is shown that when $m \geq 3$, $G(2^m, 4)$ has a k-disjoint path cover (of one-to-one type) joining any pair of two distinct source and sink for arbitrary f and $k \geq 2$ subject to $f + k \leq m$. In addition, it is proven that when $m \geq 5$, $G(2^m, 4)$ has a k-disjoint path cover (of unpaired many-to-many type) joining any two disjoint sets of k sources and k sinks for arbitrary f and $k \geq 2$ satisfying $f + k \leq m - 1$, in which sources and sinks are freely matched. In particular, the mentioned bounds $f + k \leq m$ and $f + k \leq m - 1$ of the two cases are shown to be optimal.

1. Introduction

An interconnection network is frequently modeled as a graph, where vertices and edges respectively represent nodes and communication links in the network. One of the several key problems in the study of interconnection networks is to detect (vertex-)disjoint paths that abstract the routing between nodes and the embedding of linear arrays. Such vertex-disjoint paths can be viewed as parallel routes that indicate data communication between nodes. A *k-disjoint path cover* (k-DPC for short) of a graph is a set of k internally disjoint paths that altogether cover every vertex of the graph. The k-disjoint path cover problem, originated from the community of interconnection networks, is intended to search for a way of fully utilizing nodes for efficient communications [25]. When a graph contains faulty elements, whether vertices or edges, its k-disjoint path cover naturally means a k-disjoint path cover of the graph with the faulty elements deleted.

The problem of finding such k-disjoint path covers can be classified into three kinds according to the source and sink configuration: one-to-one, one-to-many, and many-to-many. The *one-to-one* class considers disjoint path covers joining a single pair of source s and sink t, while the *one-to-many* class deals with disjoint path covers joining a single source s and a set of k distinct sinks $t_1, t_2, \ldots, t_k$. Obviously, the paths of one-to-one k-DPC, also known as k^*-container [5,28], have common vertices only at their source and sink, while those of one-to-many k-DPC overlap only at their source.

The *many-to-many* class, on the other hand, considers disjoint path covers between a set of k sources $s_1, s_2, \ldots, s_k$ and another set of k sinks $t_1, t_2, \ldots, t_k$, where any many-to-many k-DPC of graph partitions its vertex set into k paths. The

☆ This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2008-521-D00363). This work was also supported by the Catholic University of Korea, Research Fund, 2009.

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0304-3975/$ – see front matter
doi:10.1016/j.tcs.2011.04.045

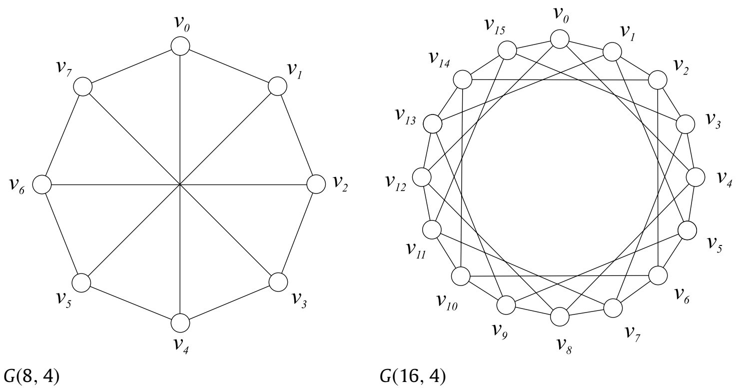

Fig. 1. Recursive circulants.

problems in this class are further subdivided into two subclasses: paired and unpaired. In the *paired* type problem, each source s_i is required to be paired to a designated sink t_i. In the *unpaired* type problem, on the other hand, the sources and sinks are allowed to be freely mapped. In other words, source s_i can be freely matched to sink t_{σ_i} under an arbitrary permutation σ on $\{1, 2, \ldots, k\}$.

Several types of graphs have already been studied on their disjoint path covers. One-to-one covers were analyzed for recursive circulants [19,28] and hypercubes with faulty edges [5]. In [20], one-to-many covers were constructed for hypercube-like interconnection networks with faulty elements. Furthermore, for a class of nonbipartite hypercube-like interconnection networks, called *restricted HL-graphs*, having faulty elements, paired disjoint path covers [25,26] and unpaired disjoint path covers [21] were built. In [13], all m-dimensional crossed cubes, twisted cubes, and Möbius cubes with $m \geq 5$ were shown to have a paired 2-DPC whose paths are of equal length.

The disjoint path cover problem has also been studied for some bipartite graphs. Paired disjoint path covers were investigated for hypercubes [11] and hypercubes with faulty vertices [8]. Unpaired disjoint path covers were considered for hypercubes with faulty edges [6] and bipartite graphs obtained by adding edges to hypercubes [7]. Interestingly, it was proven to be all NP-complete to determine if, for any fixed $k \geq 1$, there exists either a one-to-one k-DPC, a one-to-many k-DPC, or a many-to-many k-DPC, whether paired or unpaired, in general graphs [25,26].

Before turning to the next section, we briefly go over the definitions of key notions. First of all, throughout this paper, we assume that the source and sink sets S and T of graph G are disjoint to each other and both belong to $V(G)\backslash F$, where $V(G)$ and F represent the vertex set and a fault set of G, respectively. Sometimes, the sources and sinks, generally called *terminals*, are assumed to be fixed, but in our work, we deal with a stronger case where k-disjoint path covers are sought for graphs with arbitrary faulty elements and source/sink sets.

Definition 1. (a) A graph G is called *f-fault one-to-one k-disjoint path coverable* if $f + 2 \leq |V(G)|$ and for any fault set F with $|F| \leq f$, G has a one-to-one k-DPC joining an arbitrary pair of source s and sink t in $G \setminus F$ subject to $s \neq t$.
(b) A graph G is called *f-fault one-to-many k-disjoint path coverable* if $f + k + 1 \leq |V(G)|$ and for any fault set F with $|F| \leq f$, G has a one-to-many k-DPC joining an arbitrary source s and an arbitrary set T of k sinks in $G \setminus F$ subject to $s \notin T$.
(c) A graph G is called *f-fault unpaired (resp. paired) many-to-many k-disjoint path coverable* if $f + 2k \leq |V(G)|$ and for any fault set F with $|F| \leq f$, G has an unpaired (resp. paired) k-DPC joining an arbitrary set S of k sources and another arbitrary set T of k sinks in $G \setminus F$ subject to $S \cap T = \emptyset$.

This paper's interest is to investigate the construction of the disjoint path covers in recursive circulants. The recursive circulant $G(N, d)$, $d \geq 2$, proposed in [23], is a graph with a vertex set $V = \{v_0, v_1, v_2, \ldots, v_{N-1}\}$ and an edge set $E = \{(v_i, v_j) : i + d^k \equiv j \pmod N$ for some k, $0 \leq k \leq \lceil \log_d N \rceil - 1\}$. In other words, $G(N, d)$ is a circulant graph with N vertices and jumps of powers of d, $d^0, d^1, \ldots, d^{\lceil \log_d N \rceil - 1}$, which can also be defined as a Cayley graph of the cyclic group $\mathbb{Z}_N$ with the generating set $\{d^0, d^1, \ldots, d^{\lceil \log_d N \rceil - 1}\}$. Examples of $G(N, d)$ are shown in Fig. 1.

In this article, we focus on the recursive circulant $G(N, d)$ with $N = 2^m$ and $d = 4$. Such recursive circulant $G(2^m, 4)$ of degree m compares favorably to hypercube Q_m. While retaining attractive properties of the hypercube such as node-symmetry, recursive structure, maximum connectivity, etc., it achieves a noticeable improvement in diameter [23] as well as includes a complete binary tree with $2^m - 1$ vertices as a subgraph [15]. Many results on recursive circulants are found in the literature, regarding, say hamiltonian decomposition [3,10,16,18], panconnectivity and pancyclicity [1,2,22], independent spanning trees [29], maximum induced subgraph [30], chromatic number [17], parallel routing [12], recognition problem [9], edge forwarding index and bisection width [10], etc.

In the previous works, it has been shown that $G(2^m, 4)$, $m \geq 3$, is (0-fault) one-to-one k-disjoint path coverable for any $1 \leq k \leq m$ [19], is f-fault one-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + k \leq m - 1$ [20], and is f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq m$ [26]. In addition to these results, we will show that $G(2^m, 4)$, $m \geq 3$, is *f-fault* one-to-one k-disjoint path coverable for any f and $k \geq 2$ with $f + k \leq m$, and

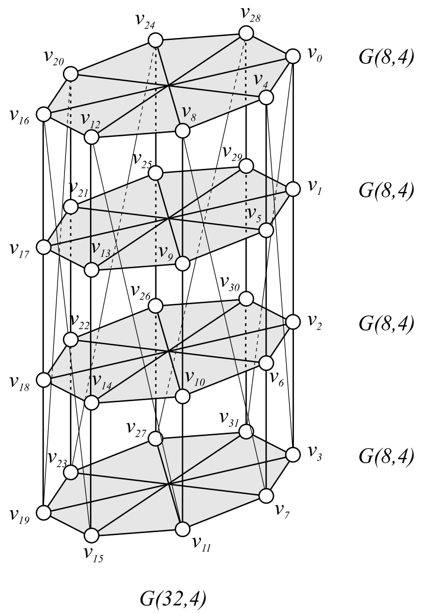

Fig. 2. Recursive structure of $G(32, 4)$.

$G(2^m, 4)$, $m \geq 5$, is f-fault *unpaired* many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + k \leq m - 1$. The bound $f + k \leq m$ achieved for a one-to-one k-DPC problem is proven optimal based upon the necessary condition shown in Lemma 7. The bound $f + k \leq m - 1$ established for an unpaired k-DPC problem is also found optimal due to the necessary condition derived in [26].

This paper is organized as follows. In the next section, we discuss the recursive structure and fault-hamiltonicity of recursive circulant and recursive circulant-like graphs. By utilizing the recursive structure, one-to-many DPCs, one-to-one DPCs, and unpaired many-to-many DPCs of recursive circulant and recursive circulant-like graphs are constructed in Sections 3–5, respectively. Finally, concluding remarks of the paper are given in Section 6.

2. Recursive structures

Before discussing the recursive structure of recursive circulants, we define a simple graph construction operation. For two graphs H_0 and H_1 with the same number of vertices, consider a bijection f between the vertex sets $V(H_0)$ and $V(H_1)$. We denote by $H_0 \oplus H_1$ the graph obtained by joining the vertices of H_0 and H_1 using edges $(v, f(v))$ for all $v \in V(H_0)$. Given $H_0 \oplus H_1$, H_0 and H_1 are called *components*, and $f(v)$ for $v \in V(H_0)$ and $f^{-1}(v)$ for $v \in V(H_1)$ are both represented by $\bar{v}$ for short.

The recursive circulant $G(N, d)$ has a recursive structure when $N = cd^m$, $1 \leq c < d$ [23], based upon the following property.

Property 1 (*[23]*)**.** *Given $G(cd^m, d)$ with $m \geq 1$, consider a vertex subset V_i such that $V_i = \{v_j : j \equiv i \pmod d\}$. Then the subgraph G_i induced by V_i is isomorphic to $G(cd^{m-1}, d)$ for all $i = 0, 1, \ldots, d-1$.*

When $m \geq 1$, $G(cd^m, d)$ can be recursively constructed using d copies of $G(cd^{m-1}, d)$, which we denote by $G_i(V_i, E_i)$, $0 \leq i < d$, with $V_i = \{v_0^i, v_1^i, \ldots, v_{cd^{m-1}-1}^i\}$. Here, G_i is isomorphic to $G(cd^{m-1}, d)$ with regard to a bijection mapping v_j^i to v_j. Let v_j^i be relabeled by v_{jd+i} for convenience. Then $G(cd^m, d)$ can be built by defining the vertex set V as $\bigcup_{0 \leq i < d} V_i$, and the edge set E as $\bigcup_{0 \leq i < d} E_i \cup X$, where $X = \{(v_j, v_{j'}) : j + 1 \equiv j' \pmod{cd^m}\}$.

Recursive circulant $G(2^m, 4)$, a special case of $G(cd^m, d)$, consists of four *components* G_0, G_1, G_2, and G_3 each of which is isomorphic to $G(2^{m-2}, 4)$ when $m \geq 2$ (see Fig. 2 to understand how $G(32, 4)$ is built from the four copies of $G(8, 4)$). It is notable that the subgraph induced by vertices in G_i and $G_{(i+1) \bmod 4}$ for any $i = 0, 1, 2, 3$, is isomorphic to the product $G(2^{m-2}, 4) \times K_2$ of $G(2^{m-2}, 4)$ and K_2, where K_2 is a complete graph with two vertices. Let H_0 and H_1 be the subgraphs induced by $V(G_0) \cup V(G_1)$ and by $V(G_2) \cup V(G_3)$, respectively. Then, the graph can be expressed as $H_0 \oplus H_1$, where H_0 and H_1 are isomorphic to $G(2^{m-2}, 4) \times K_2$.

Now, consider d copies $G_0, G_1, \ldots, G_{d-1}$ of a graph G having n vertices. If we apply the graph constructor $\oplus$ to each pair G_i and $G_{(i+1) \bmod d}$, $0 \leq i < d$, we obtain a graph with nd vertices. This graph, which is said to be obtained through the *cycle-based recursive construction*, will be denoted as $G \otimes C_d$. Here, C_d represents a cycle graph with d vertices. In the following discussion, we respectively denote by v^+ and v^- the vertices of $G_{(i+1) \bmod d}$ and $G_{(i-1) \bmod d}$ that are adjacent to v in G_i. Then, recursive circulant $G(2^m, 4)$ can also be expressed in terms of $G(2^{m-2}, 4) \otimes C_4$ as well as $[G(2^{m-2}, 4) \times K_2] \oplus [G(2^{m-2}, 4) \times K_2]$. It can be observed that any graph representable as $G \otimes C_4$ is also representable as $[G_0 \oplus G_1] \oplus [G_2 \oplus G_3]$ for some G_i's isomorphic to G, although the converse does not always hold.

In general, $G(2^m, 4)$ cannot be obtained from a single operation $\oplus$ on two recursive circulants. In other words, an arbitrary $G(2^m, 4)$ is not always representable as $H_0 \oplus H_1$ of two graphs H_0 and H_1 that are isomorphic to $G(2^{m-1}, 4)$. This implies that, when we want to recursively construct a disjoint path cover in $G(2^m, 4)$, we cannot utilize the disjoint path coverability of $G(2^{m-1}, 4)$. On the other hand, we can still utilize the disjoint path coverability of $G(2^{m-2}, 4)$, which undesirably provokes a large number of cases. Thus, we introduce a class of nonbipartite graphs containing $G(2^m, 4)$ in order to take advantage of simple recursive structure. An arbitrary higher dimensional graph (with a unique exception) may be represented as $H_0 \oplus H_1$ for two lower dimensional graphs H_0 and H_1 in the class.

Definition 2. A class of graphs, called *RC-like graphs* or *RCL-graphs* for short, is defined as follows:

- $RCL_3 = \{G(8, 4)\}$;
- $RCL_4 = \{G(16, 4), G(8, 4) \times K_2\}$;
- $RCL_m = \{G(2^m, 4), G(2^{m-1}, 4) \times K_2, G(2^{m-2}, 4) \times C_4\}$ for $m \geq 5$.

Here, a graph that belongs to RCL_m for some $m \geq 3$ is called an *m-dimensional RC-like graph*.

For convenience, we define a superclass of RC-like graphs, called the *expanded RC-like graphs*, as $RCL^e_m = \{G(2^m, 4), G(2^{m-1}, 4) \times K_2, G(2^{m-2}, 4) \times C_4\}$ for $m \geq 3$. Notice that the graph $G(4, 4) \times C_4$ in RCL^e_4 does not belong to RCL_4. Also, $G(4, 4) \times K_2$ and $G(2, 4) \times C_4$ in RCL^e_3 do not exist in RCL_3. These three graphs are bipartite, while all the graphs in the class of RC-like graphs are nonbipartite since each of them contains a subgraph isomorphic to $G(8, 4)$ or $G(16, 4)$. Now, we have a small lemma:

Lemma 1. (a) *Every RC-like graph is nonbipartite.*
(b) *Every m-dimensional RC-like graph G^m is made of 2^m vertices of degree m.*

Since each of the two graphs $G(2^m, 4)$ and $G(2^{m-2}, 4) \times C_4$ in RCL^e_m has four components G_0, G_1, G_2, and G_3, which are respectively isomorphic to $G(2^{m-2}, 4)$, they can be represented in the form of $G(2^{m-2}, 4) \otimes C_4$. Let H_0 and H_1 be the subgraphs induced by $V(G_0) \cup V(G_1)$ and by $V(G_2) \cup V(G_3)$, respectively. Then, the two graphs can also be expressed as $H_0 \oplus H_1$, where H_0 and H_1 are isomorphic to $G(2^{m-2}, 4) \times K_2$.

Let us take a look at the third graph $G(2^{m-1}, 4) \times K_2$ in RCL^e_m more carefully. It also has a recursive structure, which is derived from the recursive structure of $G(2^{m-1}, 4)$. Again, it has four components G_0, G_1, G_2, and G_3, isomorphic to $G(2^{m-3}, 4) \times K_2$. Thus, the graph can be expressed as $[G(2^{m-3}, 4) \times K_2] \otimes C_4$. If we define H_0 and H_1 as in the above paragraph, H_0 and H_1 are isomorphic to $[G(2^{m-3}, 4) \times K_2] \times K_2$, which is, in fact, isomorphic to $G(2^{m-3}, 4) \times C_4$. Therefore, the graph can also be represented as $H_0 \oplus H_1$, where H_0 and H_1 are isomorphic to $G(2^{m-3}, 4) \times C_4$. This observation leads to the next lemma.

Lemma 2. (a) *For $m \geq 5$, every m-dimensional RC-like graph G^m except for $G(16, 4) \times K_2$ can be expressed as $G \otimes C_4$, where the four components G_0, G_1, G_2, and G_3 are isomorphic to a graph G in RCL_{m-2}. Furthermore, the graph can also be expressed as $H_0 \oplus H_1$, where H_0 and H_1 are the subgraphs induced by $V(G_0) \cup V(G_1)$ and by $V(G_2) \cup V(G_3)$, respectively, and both of them are isomorphic to a graph in RCL_{m-1}.*
(b) *For $m \geq 4$, every m-dimensional RC-like graph G^m except for $G(16, 4)$ can be expressed as $H_0 \oplus H_1$, where the two components H_0 and H_1 are isomorphic to a graph in RCL_{m-1}.*

The last but not the least property of RC-like graphs we discuss in this preliminary section is the fault-hamiltonicity. A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \setminus F$ for any set F of faulty elements with $|F| \leq f$. It is worth mentioning that a graph G is f-fault (either one-to-one, one-to-many, or many-to-many) 1-disjoint path coverable if and only if G is f-fault hamiltonian-connected. In the following, let $\delta(G)$ denote the minimum degree of a graph G.

Lemma 3. *For $m \geq 3$, every m-dimensional RC-like graph is $(m-3)$-fault hamiltonian-connected and $(m-2)$-fault hamiltonian.*

Proof. It has been proven that (i) the graph $G(2^m, 4)$ with $m \geq 3$ is $(m-3)$-fault hamiltonian-connected and $(m-2)$-fault hamiltonian [24,27], and that (ii) if a graph G is $(\delta(G)-3)$-fault hamiltonian-connected and $(\delta(G)-2)$-fault hamiltonian, then $G \times K_2$ is $(\delta(G)-2)$-fault hamiltonian-connected and $(\delta(G)-1)$-fault hamiltonian [24]. Clearly, the proof of this lemma is a direct consequence of these two facts. Recall that $G \times C_4$ is isomorphic to $[G \times K_2] \times K_2$. □

3. One-to-many disjoint path covers

In this section, we will consider the problem of constructing one-to-many DPCs in RC-like graphs with faulty elements. The construction will be utilized when we build one-to-one DPCs in the graphs. The problem on recursive circulant $G(2^m, 4)$ was studied in [20] as follows.

Lemma 4 ([20]). *$G(2^m, 4)$, $m \geq 3$, is f-fault one-to-many k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq m - 1$.*

It is worthy of remark that the bound $f+k \le m-1$ achieved in Lemma 4 is optimal due to the following necessary condition given in [14]. We denote by $\kappa(G)$ the connectivity of a graph G.

Lemma 5 ([14]). *If a graph G is f-fault one-to-many k-disjoint path coverable, then $\kappa(G) \ge f+k$. Furthermore, if G has $f+k+2$ or more vertices, then $\kappa(G) \ge f+k+1$.*

To construct one-to-many DPCs in RC-like graphs, we begin by pointing out the fact in [20] that a graph G is f-fault one-to-many 2-disjoint path coverable if and only if G is f-fault one-to-many 1-disjoint path coverable, which is equivalent to that G is f-fault hamiltonian-connected. By utilizing fault-hamiltonicity of RC-like graphs given in Lemma 3, an f-fault one-to-many k-DPC for $k = 1, 2$ can be constructed when $f \le m-3$. It has been shown in [20] that an f-fault one-to-many k-DPC in $H_0 \oplus H_1$ can be recursively constructed from f-fault one-to-many $(k-1)$-DPC and fault-hamiltonicity of H_i, $i = 0, 1$, as follows.

Lemma 6 ([20]). *For $f \ge 0$ and $k \ge 3$, let H_i be a graph with n vertices satisfying the following three conditions, $i = 0, 1$.*

(a) *H_i is f-fault one-to-many $(k-1)$-disjoint path coverable.*
(b) *H_i is $(f+k-3)$-fault hamiltonian-connected (2-disjoint path coverable).*
(c) *H_i is $(f+k-2)$-fault hamiltonian.*

Then, $H_0 \oplus H_1$ is f-fault one-to-many k-disjoint path coverable.

Lemmas 3 and 6 lead to one-to-many disjoint path coverability of RC-like graphs as follows.

Theorem 1. *Every m-dimensional RC-like graph G^m, $m \ge 3$, is f-fault one-to-many k-disjoint path coverable for any f and $k \ge 2$ subject to $f+k \le m-1$.*

Proof. The proof is by induction on m. Due to Lemma 4, it suffices to consider $G(2^{m-1}, 4) \times K_2$ with $m \ge 4$ and $G(2^{m-2}, 4) \times C_4$ with $m \ge 5$. Let $H_0 \oplus H_1$ be one of these two graphs, where H_0 and H_1 are isomorphic to either $G(2^{m-1}, 4)$ or $G(2^{m-2}, 4) \times K_2$. If $k = 2$, then $f \le m-3$ and by Lemma 3, $H_0 \oplus H_1$ is f-fault one-to-many 2-disjoint path coverable. Assume $k \ge 3$. Since $f+k \le m-1$, each H_i is (i) f-fault one-to-many $(k-1)$-disjoint path coverable by induction hypothesis, (ii) $(f+k-3)$-fault hamiltonian-connected by Lemma 3, and (iii) $(f+k-2)$-fault hamiltonian by Lemma 3. Thus, by Lemma 6, $H_0 \oplus H_1$ is f-fault one-to-many k-disjoint path coverable. This completes the proof. □

Of course, the bound $f+k \le m-1$ achieved in Theorem 1 is optimal due to Lemma 5.

4. One-to-one disjoint path covers

We begin with a necessary condition for a graph to be f-fault one-to-one k-disjoint path coverable.

Lemma 7. *If a graph G is f-fault one-to-one k-disjoint path coverable, then $\kappa(G) \ge f+k$.*

Proof. According to Menger's theorem (see Ref. [4]), a graph G is k-connected if and only if for every pair of source s and sink t, G has k internally disjoint paths of type one-to-one joining them. A one-to-one k-disjoint path coverable graph should be k-connected, and thus the lemma follows. □

We are going to construct f-fault one-to-one k-disjoint path covers in m-dimensional RC-like graphs for any f and $k \ge 2$ satisfying the optimal bound $f+k \le m$ of Lemma 7. That is, we will establish the following theorem.

Theorem 2. *Every m-dimensional RC-like graph G^m, $m \ge 3$, is f-fault one-to-one k-disjoint path coverable for any f and $k \ge 2$ subject to $f+k \le m$.*

A graph G is f-fault one-to-one 2-disjoint path coverable if and only if G is f-fault hamiltonian. Thus, to prove Theorem 2, we can assume that

$$k \ge 3$$

due to Lemma 3. A path in a graph is represented as a sequence of vertices. An *s-t path* refers to a path from vertex s to t, and an *s-path* refers to a path whose starting vertex is s.

4.1. Proof of Theorem 2 when $f = 0$

The one-to-one DPC problem in fault-free $G(2^m, 4)$ was studied in [19] as follows. We denote by $P(l)$ a graph isomorphic to a path having l vertices. In $G(2^{m-2}, 4) \times P(l)$ with $l \ge 2$, each *component* is isomorphic to $G(2^{m-2}, 4)$ and referred to $G_0, G_1, \ldots, G_{l-1}$.

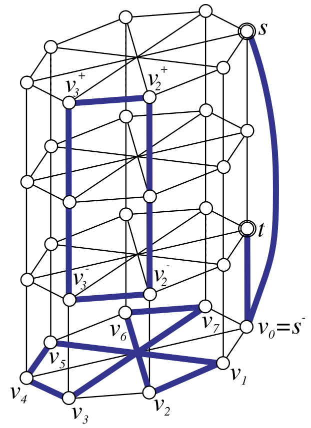

Fig. 3. Illustration of the proof of Lemma 10.

Lemma 8 ([19]). (a) *$G(2^m, 4)$ with $m \geq 3$ is one-to-one k-disjoint path coverable for any $1 \leq k \leq m$.*
(b) *$G(2^{m-1}, 4) \times P(l)$ with $m \geq 4$ and $l \geq 2$ has a one-to-one k-DPC joining any source s in G_0 and sink t in G_{l-1} for any $1 \leq k \leq m$.*

Now, let us consider one-to-one disjoint path coverability of $G(2^{m-1}, 4) \times K_2$ and $G(2^{m-2}, 4) \times C_4$.

Lemma 9. *$G(2^{m-1}, 4) \times K_2$ with $m \geq 4$ is one-to-one k-disjoint path coverable for any $3 \leq k \leq m$.*

Proof. Let G_0 and G_1 be components isomorphic to $G(2^{m-1}, 4)$. If $s \in V(G_0)$ and $t \in V(G_1)$, then by Lemma 8(b), there exists a one-to-one k-DPC joining s and t. Now let $s, t \in V(G_0)$. We first construct a one-to-one $(k-1)$-DPC in G_0 by Lemma 8(a), and then path $P_k = (s, \bar{s}, P_h, \bar{t}, t)$ is added to the DPC, where P_h is an $\bar{s}$-$\bar{t}$ hamiltonian path in G_1. Thus, we have the lemma. $\square$

Lemma 10. *$G(2^{m-2}, 4) \times C_4$ with $m \geq 5$ is one-to-one k-disjoint path coverable for any $3 \leq k \leq m$.*

Proof. Let G_0, G_1, G_2, and G_3 be the four components isomorphic to $G(2^{m-2}, 4)$. We assume $s \in V(G_0)$ and let $t \in V(G_i)$. We assume w.l.o.g. $i = 0, 1$, or 2. If $i = 0$, we first find a one-to-one $(k-2)$-DPC in G_0, and then add two paths $P_{k-1} = (s, s^+, P_h^1, t^+, t)$ and $P_k = (s, s^-, P_h^2, t^-, t)$ to the DPC, where P_h^1 is a hamiltonian path in G_1 joining s^+ and t^+, and P_h^2 is a hamiltonian path in the subgraph H_1 induced by $V(G_2) \cup V(G_3)$ joining s^- and t^-. It can be easily seen that H_1 is hamiltonian-connected since both G_2 and G_3 are hamiltonian-connected. If $i = 1, 2$, we first find a one-to-one $(k-1)$-DPC in the subgraph induced by $V(G_0) \cup \cdots \cup V(G_i)$ joining s and t by Lemma 8(b). For the subcase of either $i = 1$ or $i = 2$ and $s^- \neq t^+$, we add path $P_k = (s, s^-, P_h, t^+, t)$ to the DPC, where P_h is a hamiltonian path in the subgraph induced $V(G_{i+1}) \cup \cdots \cup V(G_3)$ joining s^- and t^+.

Finally, let $i = 2$ and $s^- = t^+$. In this subcase, let the last path $P_k = (s, s^-, t)$. To cover the vertices in G_3 other than s^-, we are going to pick up an edge $(x, y) \in E(G_2) \cup E(G_0)$ on some path P_j in the DPC such that $G_3 \setminus s^-$ has a hamiltonian path P_h joining x^+ and y^+ when $x, y \in V(G_2)$ or joining x^- and y^- when $x, y \in V(G_0)$. And then, the edge (x, y) on P_j is replaced with (x, x^+, P_h, y^+, y) or (x, x^-, P_h, y^-, y), resulting in a new path P_j'. If $m \geq 6$, an arbitrarily edge (x, y) in G_2 or in G_0 such that $\{x, y\} \cap \{s, t\} = \emptyset$ is acceptable since G_3 is $(m-5)$-fault hamiltonian-connected. Let $m = 5$. G_3 is 1-fault hamiltonian and thus $G_3 \setminus s^-$ has a hamiltonian cycle, say $C_h = (v_1, v_2, v_6, v_7, v_3, v_4, v_5)$ assuming $s^- = v_0$. It suffices to show that for some edge (a, b) on C_h, at least one of (a^-, b^-) and (a^+, b^+) is passed through by some path in the $(k-1)$-DPC. Suppose, for a contradiction, that no such edge exists. See Fig. 3. None of the edges (v_1^-, v_2^-), (v_2^-, v_6^-), (v_3^-, v_4^-), and (v_3^-, v_7^-) is passed through by any path, and thus path segment $R_1 = ((v_2^-)^-, v_2^-, v_3^-, (v_3^-)^-)$ must be passed through by some path in the DPC. Similarly, we observe that path segment $R_2 = ((v_2^+)^+, v_2^+, v_3^+, (v_3^+)^+)$ must be passed through by some path. The two path segments R_1 and R_2 form a cycle of length six, which is a contradiction to the fact that the path segments must be passed through by some paths in the DPC. This completes the proof. $\square$

4.2. Proof of Theorem 2 when $f \geq 1$

It has been known in [25] that an f-fault one-to-many k-disjoint path coverable graph is always f-fault one-to-one k-disjoint path coverable. To prove Theorem 2, due to Theorem 1, it can be assumed that

$$f + k = m.$$

Since $k \geq 3$ and $f \geq 1$, we have $m \geq 4$. Furthermore, we assume

$$(s, t) \notin E(G^m) \setminus F.$$

Suppose otherwise. Then, regarding (s, t) as a virtual fault allows us to find $(f + 1)$-fault one-to-one $(k - 1)$-DPC and to add the path (s, t) to the DPC, resulting in an f-fault k-DPC. We also assume that when $f = 1$ and $k = m - 1$,

$$F \neq \{(s, t)\} \text{ and } F \neq \{v_f\} \text{ for any } v_f \text{ with } (s, v_f), (t, v_f) \in E(G^m).$$

Suppose otherwise. Then, regarding the faulty element as a virtual fault-free element allows us to find a 0-fault one-to-one m-DPC (by the algorithm in Section 4.1) and to remove the path either (s, t) or (s, v_f, t) passing through the faulty element from the DPC, resulting in a 1-fault $(m - 1)$-DPC.

The proof will proceed by induction on m. Recall that every m-dimensional RC-like graph G^m, $m \geq 4$, except for $G(16, 4)$ can be expressed as $H_0 \oplus H_1$, where $H_0, H_1 \in RCL_{m-1}$ by Lemma 2(b). To construct an f-fault one-to-one k-DPC in G^m, the recursive structure of $H_0 \oplus H_1$ will be utilized. For the exception $G(16, 4)$, a computer program for finding 1-fault 3-DPC for given a fault and a pair of s and t was written in C language. The validity of the following lemma was checked by the program.

Lemma 11. *$G(16, 4)$ is 1-fault one-to-one 3-disjoint path coverable.*

From now on, let G^m be expressed as $H_0 \oplus H_1$. F_0 and F_1 denote the sets of faulty elements in H_0 and H_1, respectively, and F_2 denotes the set of faulty edges joining vertices in H_0 and vertices in H_1, so that $F = F_0 \cup F_1 \cup F_2$. Let $f_0 = |F_0|, f_1 = |F_1|$, and $f_2 = |F_2|$. Since a one-to-one k-DPC in $H_0 \oplus H_1$ with a virtual fault set $F \cup F'$, where F' is a set of arbitrary $f - |F|$ fault-free edges, is also a one-to-one k-DPC in $H_0 \oplus H_1$ with the fault set F, we assume $|F| = f$.

Remember that each H_i is $(m - 4)$-fault hamiltonian-connected and $(m - 3)$-fault hamiltonian by Lemma 3. A vertex v is called *free* if v is fault-free and not a terminal. An edge (v, w) is called *free* if v and w are free and $(v, w) \notin F$. There are three cases.

Case 1: $s, t \in V(H_0)$ and $f_1 + f_2 = 0$ $(f_0 = f)$.

We first present a procedure for constructing a one-to-one DPC for this case, and then show that the procedure is correct.

Procedure DPC-A$(H_0 \oplus H_1, s, t, F)$
/* $s, t \in V(H_0)$ and $f_1 + f_2 = 0$ $(f_0 = f)$. */

1. Regarding a faulty element α as a *virtual* fault-free element, find an $(f_0 - 1)$-fault k-DPC in H_0.
2. When some path P_i in the DPC passes through α, let $P_i = (s, P_s, x, \alpha, y, P_t, t)$ if α is a vertex; let $P_i = (s, P_s, x, y, P_t, t)$ if α is an edge (x, y). Here, P_s and P_t are path segments of P_i. When no path in the DPC passes through α, pick up an arbitrarily path $P_i = (s, P_s, x, y, P_t, t)$ in the DPC.
3. Replace P_i with $P'_i = (s, P_s, x, \bar{x}, P_h, \bar{y}, y, P_t, t)$, where P_h is a hamiltonian path in H_1 between $\bar{x}$ and $\bar{y}$.

Lemma 12. *When $s, t \in V(H_0)$ and $f_1 + f_2 = 0$ $(f_0 = f)$, Procedure DPC-A constructs an f-fault one-to-one k-DPC for any $m \geq 4$.*

Proof. The $(f_0 - 1)$-fault k-DPC in Step 1 exists since $(f_0 - 1) + k = f + k - 1 = m - 1$. The $\bar{x}$-$\bar{y}$ hamiltonian path in Step 3 exists due to Lemma 3. Thus, Procedure DPC-A can always be applied. □

Case 2: $s \in V(H_0)$ and $t \in V(H_1)$.

In this case, it is assumed that $f_0 \geq f_1$.

Procedure DPC-B$(H_0 \oplus H_1, s, t, F)$
/* $s \in V(H_0)$, $t \in V(H_1)$, and $f_0 \geq f_1$. */

1. Let $z = \bar{t}$ if $(t, \bar{t}), \bar{t} \notin F$; otherwise, let z be a free vertex in H_0 such that $(z, \bar{z}), \bar{z} \notin F$. Find an f_0-fault s-z hamiltonian path in H_0.
2. Pick up $k - 1$ distinct vertices $z_1, z_2, \ldots, z_{k-1}$ on the hamiltonian path such that for each i, $(s, z_i) \in E(G^m) \setminus F$ and $\bar{x}_i, (x_i, \bar{x}_i) \notin F$, where x_i is the vertex on the hamiltonian path that precedes z_i.
3. If $z = \bar{t}$, find f_1-fault *one-to-many* $(k - 1)$-DPC in H_1 joining $\{\bar{x_1}, \bar{x_2}, \ldots, \bar{x_{k-1}}\}$ and t; if $z \neq \bar{t}$, find f_1-fault *one-to-many* k-DPC in H_1 joining $\{\bar{x_1}, \bar{x_2}, \ldots, \bar{x_{k-1}}, \bar{z}\}$ and t;
4. Merge the hamiltonian path and the one-to-many DPC with edges $(z, \bar{z})$ and $(x_i, \bar{x}_i)$, $1 \leq i \leq k - 1$. Discard edges (x_i, z_i) for all i with $x_i \neq s$.

Lemma 13. *When $s \in V(H_0)$ and $t \in V(H_1)$, Procedure DPC-B constructs an f-fault one-to-one k-DPC for any $m \geq 4$ unless* (a) *$k = 3$ and $f_0 = m - 3$ or* (b) *$f_0 + f_2 = 1$ and one of $\bar{t}$ or $(t, \bar{t})$ is faulty.*

Proof. The hamiltonian path in Step 1 exists if $f_0 \leq m - 4$. Thus, it exists unless $k = 3$ and $f_1 + f_2 = 0$ $(f_0 = m - 3)$ since $f_0 = f - (f_1 + f_2) = m - k - (f_1 + f_2)$. Notice $k \geq 3$. All the vertices adjacent to s are candidates for z_i's of Step 2. Each faulty element may block at most one candidate. There are $m - 1$ candidates and at most f blocking elements, and thus the number of nonblocked candidates is at least $m - 1 - f = k - 1$. Thus, we can always pick up $k - 1$ vertices $z_1, z_2, \ldots, z_{k-1}$ on the hamiltonian path. When $z = \bar{t}$, the f_1-fault one-to-many $(k - 1)$-DPC in Step 3 exists if $f_1 + (k - 1) \leq m - 2$ by Theorem 1. By the assumption of $f \geq 1$, we have $f_0 + f_2 \geq 1$. Recall $f_0 \geq f_1$. Then, $f_1 + (k - 1) = f - (f_0 + f_2) + (k - 1) = m - (f_0 + f_2) - 1 \leq m - 2$. Similarly, we can see that when $z \neq \bar{t}$ ($\bar{t}$ or $(t, \bar{t})$ is faulty), the f_1-fault one-to-many k-DPC in Step 3 exists unless $f_0 + f_2 = 1$. This completes the proof. □

The two exceptional cases (a) and (b) of Lemma 13 are considered in the following two lemmas.

Lemma 14. *When $s \in V(H_0)$, $t \in V(H_1)$, $k = 3$, and $f_0 = m - 3$, there exists an f-fault one-to-one k-DPC in $H_0 \oplus H_1$ for any $m \geq 4$.*

Proof. There exists a hamiltonian cycle C_h in $H_0 \setminus F_0$. When $m \geq 5$, let (x, y) be an edge on C_h such that $x, y \neq s$ and $\bar{x}, \bar{y} \neq t$. A one-to-many 3-DPC in H_1 joining $\{\bar{s}, \bar{x}, \bar{y}\}$ and t is merged with C_h to obtain a one-to-one 3-DPC in $H_0 \oplus H_1$. Let $m = 4$ and $H_0 \oplus H_1$ be isomorphic to $G(8, 4) \times K_2$. If $\bar{t} \notin F$, we pick up an edge (x, y) on C_h such that $x = \bar{t}$ and $y \neq s$. A one-to-many 2-DPC in H_1 between $\{\bar{s}, \bar{y}\}$ and t is merged with C_h for our purpose. The last subcase of $\bar{t} \in F$ is deferred to Lemma 33 in Appendix. □

Lemma 15. *When $s \in V(H_0)$, $t \in V(H_1)$, $f_0 + f_2 = 1$, and one of $\bar{t}$ or $(t, \bar{t})$ is faulty, there exists an f-fault one-to-one k-DPC in $H_0 \oplus H_1$ for any $m \geq 4$.*

Proof. If $(t, \bar{t})$ is faulty, then $f_0 = 0$ and thus $F = \{(t, \bar{t})\}$. By the assumption of $F \neq \{(s, t)\}$, we have $\bar{t} \neq s$ and $(s, \bar{s}) \notin F$. It suffices to switch H_0 and H_1 and apply Procedure DPC-B($H_1 \oplus H_0, t, s, F$). Let $\bar{t}$ be faulty. If $f_1 = 1$ and $\bar{s} \notin F$, similar to the previous case, it suffices to switch H_0 and H_1 and apply Procedure DPC-B($H_1 \oplus H_0, t, s, F$). Hereafter in this proof, we assume $F = \{\bar{t}\}$ or $F = \{\bar{t}, \bar{s}\}$. When (i) $m \geq 6$ or (ii) $m = 5$ and $H_0 \oplus H_1$ is isomorphic to $G(32, 4)$ or $G(8, 4) \times C_4$, we let G_0, G_1, G_2, and G_3 be the four components of the graph such that H_0 and H_1 are the subgraphs induced by $V(G_0) \cup V(G_1)$ and by $V(G_2) \cup V(G_3)$, respectively. Note that all G_i's are isomorphic to a graph in RCL_{m-2}, and that the subgraph induced by $V(G_i) \cup V(G_{(i+1) \bmod 4})$ for each $i = 0, 1, 2, 3$, is isomorphic to a graph in RCL_{m-1}. Assume w.l.o.g. $t \in V(G_2)$. Then, $\bar{t} \in V(G_1)$.

If $s \in V(G_1)$, it suffices to apply Procedure DPC-A($H'_0 \oplus H'_1, s, t, F$), where H'_0 and H'_1 are the subgraphs induced by $V(G_1) \cup V(G_2)$ and by $V(G_3) \cup V(G_0)$, respectively. Let $s \in V(G_0)$. If $(s, \bar{t}) \notin E(G^m)$, then it suffices to apply Procedure DPC-B($H'_0 \oplus H'_1, t, s, F$). When $F = \{\bar{t}\}$, by the assumption of $F \neq \{v_f\}$ for any v_f with $(s, v_f), (t, v_f) \in E(G^m)$, we always have $(s, \bar{t}) \notin E(G^m)$ and thus we are done. If $F = \{\bar{t}, \bar{s}\}$ and $(t, \bar{s}) \notin E(G^m)$, it suffices to apply Procedure DPC-B($H'_1 \oplus H'_0$, s, t, F). Finally, if $F = \{\bar{t}, \bar{s}\}$ and $(s, \bar{t}), (t, \bar{s}) \in E(G^m)$, then regarding $\bar{s}$ and $\bar{t}$ as virtual fault-free vertices, it suffices to find 0-fault one-to-one m-DPC and remove the two paths $(s, \bar{s}, t)$ and $(s, \bar{t}, t)$ from the DPC.

The case when $m = 5$ and $H_0 \oplus H_1$ is isomorphic to $G(16, 4) \times K_2$ is deferred to Lemma 34 in Appendix. The case when $m = 4$ and $H_0 \oplus H_1$ is isomorphic to $G(8, 4) \times K_2$ is also deferred to Lemma 33. This completes the proof. □

Case 3: $s, t \in V(H_0)$ and $f_1 + f_2 \geq 1$.

Procedure DPC-C($H_0 \oplus H_1, s, t, F$)
/* $s, t \in V(H_0)$ and $f_1 + f_2 \geq 1$. */

1. Find an f_0-fault one-to-one k-DPC in H_0.
2. For some edge (x, y) on a path P_i in the DPC such that x, $(x, \bar{x})$, y, and $(y, \bar{y})$ are all fault-free, (x, y) is replaced with $(x, \bar{x}, P_h, \bar{y}, y)$, where P_h is a hamiltonian path in $G_1 \setminus F_1$ between $\bar{x}$ and $\bar{y}$.

Lemma 16. *When $s, t \in V(H_0)$ and $f_1 + f_2 \geq 1$, Procedure DPC-C constructs an f-fault one-to-one k-DPC for any $m \geq 4$ unless $k = 3$ and $f_1 = m - 3$.*

Proof. The f_0-fault one-to-one k-DPC in Step 1 exists since $f_0 + k = f - (f_1 + f_2) + k = m - (f_1 + f_2) \leq m - 1$. The $\bar{x}$-$\bar{y}$ hamiltonian path in Step 2 exists if $f_1 = f - (f_0 + f_2) = m - k - (f_0 + f_2) \leq m - 4$. That is, it exists unless $k = 3$ and $f_0 = f_2 = 0$ ($f_1 = m - 3$). Thus, we have the lemma. □

Lemma 17. *When $s, t \in V(H_0)$, $k = 3$, and $f_1 = m - 3$, there exists an f-fault one-to-one k-DPC in $H_0 \oplus H_1$ for any $m \geq 4$.*

Proof. Let us consider the case $m \geq 6$ first. There exists a free vertex x in H_0 adjacent to s such that $\bar{x} \notin F$. Since H_1 is $(m - 3)$-fault hamiltonian, there exists a fault-free vertex y in H_1 such that $\bar{y} \neq s$ and $\bar{x}$ and y are joined by a hamiltonian path in $H_1 \setminus F_1$. Let z be a free vertex in H_0 adjacent to s such that $z \neq x, \bar{y}$. Regarding x as a *virtual* fault, we find a one-to-many 3-DPC joining $\{s, z, \bar{y}\}$ and t if $\bar{y} \neq t$; otherwise, we find a one-to-many 2-DPC joining $\{s, z\}$ and t. The one-to-many DPC in H_0 and the hamiltonian path in $H_1 \setminus F_1$ are merged with edges (s, z), (s, x), $(x, \bar{x})$, and $(y, \bar{y})$, resulting in a desired one-to-one 3-DPC joining s and t.

Second, let $m = 5$ and $F \neq \{\bar{s}, \bar{t}\}$. Assume $\bar{s} \notin F$. Similar to the case $m \geq 6$, a one-to-one 3-DPC can be obtained by merging a hamiltonian path in $H_1 \setminus F_1$ between $\bar{s}$ and a fault-free vertex y such that $\bar{y} \neq t$ and a one-to-many 3-DPC in H_0 between $\{s, z, \bar{y}\}$ and t, where z is a free vertex adjacent to s in H_0 such that $z \neq \bar{y}$. Now, let $m = 5$, $F = \{\bar{s}, \bar{t}\}$, and $H_0 \oplus H_1$ be isomorphic to $G(32, 4)$ or $G(8, 4) \times C_4$. As in the proof of Lemma 15, this graph has four components G_0, G_1, G_2, and G_3 such that H_0 and H_1 are the subgraphs induced by $V(G_0) \cup V(G_1)$ and by $V(G_2) \cup V(G_3)$, respectively. If both s and t are contained in the same component, say G_1, it suffices to apply Procedure DPC-A($H'_0 \oplus H'_1, s, t, F$), where H'_0 and H'_1 are the subgraphs induced by $V(G_1) \cup V(G_2)$ and by $V(G_3) \cup V(G_0)$, respectively. If s and t are contained in different components, say $s \in V(G_1)$ and $t \in V(G_0)$, it suffices to apply Procedure DPC-B($H'_0 \oplus H'_1, s, t, F$). The case when $m = 5$, $F = \{\bar{s}, \bar{t}\}$, and $H_0 \oplus H_1$ is isomorphic to $G(16, 4) \times K_2$ is deferred to Lemma 32 in Appendix. The last case of $m = 4$ is also deferred to Lemma 31. □

5. Unpaired many-to-many disjoint path covers

In terms of connectivity and the minimum degree, necessary conditions for a graph to be f-fault unpaired many-to-many k-disjoint path coverable were derived in [26] as follows.

Lemma 18 (*[26]*). *Let G be an f-fault unpaired many-to-many $k(\geq 2)$-disjoint path coverable graph. Then, $\kappa(G) \geq f + k$. Furthermore, if G has $f + 2k + 1$ or more vertices, then $\delta(G) \geq f + k + 1$.*

In this section, we will construct f-fault unpaired k-disjoint path covers in m-dimensional RC-like graphs with $m \geq 5$ for any f and $k \geq 2$ satisfying the optimal bound $f + k \leq m - 1$ given in Lemma 18. That is, we will establish the following theorem.

Theorem 3. *Every m-dimensional RC-like graph G^m, $m \geq 5$, is f-fault unpaired many-to-many k-disjoint path coverable for any $f \geq 0$ and $k \geq 2$ subject to $f + k \leq m - 1$.*

The 4-dimensional RC-like graphs are not 0-fault unpaired 3-disjoint path coverable. However, they are 0-fault unpaired 2-disjoint path coverable, which is a direct consequence of a result in [26] that every m-dimensional RC-like graph, $m \geq 4$, is f-fault *paired* many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq m$. Notice that a paired many-to-many k-disjoint path coverable graph is always unpaired k-disjoint path coverable.

Lemma 19. *Every G^4 is 0-fault unpaired 2-disjoint path coverable.*

The proof of Theorem 3 will proceed by induction on m. For the base case of $m = 5$, we obtained the following Lemma 20 from a computer program that exhaustively searched out f-fault unpaired k-DPCs for any $f \geq 0$ and $k \geq 2$ satisfying $f + k \leq 4$.

Lemma 20. *Every G^5 is f-fault unpaired k-disjoint path coverable for any $f \geq 0$ and $k \geq 2$ with $f + k \leq 4$.*

Let $m \geq 6$, and recall that G^m is isomorphic to $H_0 \oplus H_1$ for some $H_0, H_1 \in RCL_{m-1}$. We will construct an f-fault unpaired k-DPC for any given set S of k sources and set T of k sinks in G^m having at most f faulty elements such that $f + k \leq m - 1$. An unpaired k-DPC with a fault set F is also an unpaired k-DPC with a virtual fault set $F \cup F'$, where F' is a set of arbitrary $m - 1 - k - |F|$ fault-free edges. As a result, it can be assumed that

$$f = |F| \quad \text{and} \quad f + k = m - 1.$$

We denote by S_i and T_i the sets of sources and sinks in H_i, $i = 0, 1$, respectively. We assume w.l.o.g. that $|S_0| \geq |T_0|$ and $|S_1| \leq |T_1|$. We let $k_0 = |T_0|$, $k_1 = |S_1|$, and $k_2 = k - (k_0 + k_1)$. Then, H_0 has $k_0 + k_2$ sources and k_0 sinks, and H_1 has k_1 sources and $k_1 + k_2$ sinks. We assume that $S_0 = \{s_i : 1 \leq i \leq k_0 + k_2\}$, $S_1 = \{s_i : k_0 + k_2 < i \leq k\}$, $T_0 = \{t_j : 1 \leq j \leq k_0\}$, and $T_1 = \{t_j : k_0 < j \leq k\}$. Furthermore, we also assume w.l.o.g. that

$$k_0 \geq k_1, \quad \text{and if } k_0 = k_1, f_0 \geq f_1.$$

Hereafter in this section, an unpaired k-DPC in a graph G with fault set F joining S and T is denoted by k-DPC$[S, T|G, F]$. We have three cases. Remember $k \geq 2$.

Case 1: $k_1 \geq 1$ or $f_0 \leq f - 1$.

We first present a basic procedure for constructing an unpaired DPC in this case.

Procedure DPC-D$(H_0 \oplus H_1, S, T, F)$
/* $k_1 \geq 1$ or $f_0 \leq f - 1$. */

1. Pick up k_2 free edges joining vertices in H_0 and vertices in H_1. Let X_0 be the set of endvertices of the free edges in H_0 and X_1 be in H_1.
2. Find a $(k_0 + k_2)$-DPC$[S_0, T_0 \cup X_0|H_0, F_0]$.
3. Case $k_1 + k_2 \geq 1$:
   (a) Find a $(k_1 + k_2)$-DPC$[S_1 \cup X_1, T_1|H_1, F_1]$.
   (b) Merge the two DPCs with the k_2 free edges.
4. Case $k_1 + k_2 = 0$:
   (a) Let (x, y) be an edge on some path in the $(k_0 + k_2)$-DPC such that all the $\bar{x}$, $(x, \bar{x})$, $\bar{y}$, and $(y, \bar{y})$ are fault-free.
   (b) Find a hamiltonian path joining $\bar{x}$ and $\bar{y}$ in $H_1 \setminus F_1$.
   (c) Merge the $(k_0 + k_2)$-DPC and the hamiltonian path with the edges $(x, \bar{x})$ and $(y, \bar{y})$. Discard the edge (x, y).

Lemma 21. *When $k_1 \geq 1$ or $f_0 \leq f - 1$, Procedure DPC-D constructs an f-fault unpaired k-DPC for any $m \geq 6$ unless* (a) $k_0 = 1$, $k_1 = 1$, *and* $f_0 = m - 3$, (b) $k_0 = 1$, $k_2 = 1$, *and* $f_1 = m - 3$, *or* (c) $k_0 = 2$ *and* $f_1 = m - 3$.

Proof. For Step 1, we have 2^{m-1} candidate edges and $f + 2k$ blocking elements (f faults and $2k$ terminals). The number of nonblocked candidates is at least $2^{m-1} - (f + 2k) \geq 2^{m-1} - 2(m-1) > m > k_2$ for any $m \geq 6$. Thus, it is possible to pick up k_2 free edges. Since $f_0 + (k_0 + k_2) = f_0 + (k - k_1) \leq f + k - 1 = m - 2$, by induction hypothesis, the $(k_0 + k_2)$-DPC in Step 2 exists when $k_0 + k_2 \geq 2$. If $k_0 + k_2 = 1$, the $(k_0 + k_2)$-DPC is indeed a hamiltonian path, and it exists, by Lemma 3, when $f_0 \leq m - 4$. Thus, the $(k_0 + k_2)$-DPC in Step 2 exists unless $f_0 = m - 3$ $(k = 2)$ and $k_0 + k_2 = 1$ $(k_1 = 1)$, or equivalently, unless the exceptional case (a). For Step 3, note that $f_1 + (k_1 + k_2) = f_1 + (k - k_0) \leq f + k - 1 = m - 2$. Recall the assumption that $k_0 \geq k_1$, and that if $k_0 = k_1, f_0 \geq f_1$. If $k_1 + k_2 \geq 2$, the $(k_1 + k_2)$-DPC exists. If $k_1 + k_2 = 1$, the $(k_1 + k_2)$-DPC exists unless $f_1 = m - 3$. Thus, the $(k_1 + k_2)$-DPC in Step 3 exists unless $f_1 = m - 3$ $(k = 2)$ and $k_1 + k_2 = 1$ $(k_0 = 1)$, i.e., unless the exceptional case (b). Finally, the hamiltonian path in Step 4(b) exists unless $f_1 = m - 3$. That is, it exists unless $f_1 = m - 3$ $(k = 2)$ and $k_1 + k_2 = 0$ $(k_0 = 2)$, i.e., unless the exceptional case (c). This completes the proof. □

The three exceptional cases (a), (b), and (c) of Lemma 21 are considered in the following three lemmas.

Lemma 22. *When $k_0 = 1$, $k_1 = 1$, and $f_0 = m - 3$, there exists an f-fault unpaired k-DPC in $H_0 \oplus H_1$ for any $m \geq 6$.*

Proof. There exists a hamiltonian cycle C_h in $H_0 \setminus F_0$ by Lemma 3. Let $C_h = (s_1, P_a, t_1, P_b)$ for some subpaths P_a and P_b. We assume w.l.o.g. the length of P_a is at least that of P_b. Let $P_a = (x, P'_a, y)$. Then, $x \neq y$. If $\{\bar{x}, \bar{y}\} \cap \{s_2, t_2\} = \emptyset$, it suffices to find 2-DPC$[\{s_2, t_2\}, \{\bar{x}, \bar{y}\}|H_1, \emptyset]$ and merge C_h and the 2-DPC with edges $(x, \bar{x})$ and $(y, \bar{y})$. Of course, we discard the edges (s_1, x) and (t_1, y). If $|\{\bar{x}, \bar{y}\} \cap \{s_2, t_2\}| = 1$, say $\bar{x} = s_2$, it suffices to find a $\bar{y}$-t_2 hamiltonian path P_h in $H_1 \setminus s_2$ and then merge C_h and P_h with $(x, \bar{x})$ and $(y, \bar{y})$. Finally in case $\{\bar{x}, \bar{y}\} = \{s_2, t_2\}$, let subpath $(t_1, P_b) = (P'_b, z)$. It suffices to find a $\bar{s_1}$-$\bar{z}$ hamiltonian path P'_h in $H_1 \setminus \{s_2, t_2\}$ and merge C_h and P'_h with edges $(s_1, \bar{s_1})$ and $(z, \bar{z})$. The existence of P'_h is due to Lemma 3. The proof is completed. □

Lemma 23. *When $k_0 = 1$, $k_2 = 1$, and $f_1 = m - 3$, there exists an f-fault unpaired k-DPC in $H_0 \oplus H_1$ for any $m \geq 6$.*

Proof. There exists a hamiltonian cycle C_h in $H_1 \setminus F_1$, and let $C_h = (t_2, x, P_a, y)$ for some subpath P_a. Then, $\bar{x} \neq t_1$ or $\bar{y} \neq t_1$. Assume $\bar{y} \neq t_1$. If $\bar{y} \notin \{s_1, s_2\}$, it suffices to find 2-DPC$[\{s_1, s_2\}, \{t_1, \bar{y}\}|H_0, \emptyset]$ and merge the 2-DPC and C_h with edge $(\bar{y}, y)$. If $\bar{y} \in \{s_1, s_2\}$, say $\bar{y} = s_2$, it suffices to find an s_1-t_1 hamiltonian path P_h in $H_0 \setminus s_2$ and merge P_h and C_h with $(\bar{y}, y)$. Thus, we have the lemma. □

Lemma 24. *When $k_0 = 2$ and $f_1 = m - 3$, there exists an f-fault unpaired k-DPC in $H_0 \oplus H_1$ for any $m \geq 6$.*

Proof. We consider the first case that for some terminal, say s_1, $\bar{s_1}$ is fault-free. There exists a hamiltonian cycle C_h in $H_1 \setminus F_1$, and let $C_h = (\bar{s_1}, x, P_a, y)$ for some subpath P_a. Assume w.l.o.g. $\bar{y} \neq s_2$. If $\bar{y} \notin \{t_1, t_2\}$, it suffices find 2-DPC$[\{\bar{y}, s_2\}, \{t_1, t_2\}|H_0, \{s_1\}]$ and merge the 2-DPC and C_h with edge $(s_1, \bar{s_1})$ and $(\bar{y}, y)$. If $\bar{y} \in \{t_1, t_2\}$, say $\bar{y} = t_1$, we find an s_2-t_2 hamiltonian path in $H_0 \setminus \{s_1, t_1\}$ and let s_1-t_1 path be $(s_1, C_h \setminus (\bar{s_1}, y), t_1)$. For the second case, we assume that $\bar{s_1}, \bar{t_1}, \bar{s_2}$, and $\bar{t_2}$ are all faulty. This implies $f_1 \geq 4$ and thus $m \geq 7$. We claim that there exists a free edge $(x, \bar{x})$ with $x \in V(H_0)$ such that x is adjacent to s_1. There are $m - 1$ candidate edges. The number of blocking elements is at most $m - 3$ since $\bar{s_2}$, $\bar{t_1}$, and $\bar{t_2}$ are all faulty. Thus, the claim is proved. Let a hamiltonian cycle C_h in $H_1 \setminus F_1$ be $(\bar{x}, w, P_b, z)$. Since $\bar{z} \notin \{s_1, s_2, t_1, t_2\}$, it suffices to find 2-DPC$[\{\bar{z}, s_2\}, \{t_1, t_2\}|H_0, \{s_1, x\}]$ and merge the 2-DPC and C_h with edges (s_1, x), $(x, \bar{x})$, and $(\bar{z}, z)$. This completes the proof. □

Case 2: $k_1 = 0, f_0 = f$, and $k_0 \geq 1$ or $f_0 \geq 1$.

We present two basic Procedures DPC-E and DPC-F depending on whether $k_0 = k$ or not.

Procedure DPC-E$(H_0 \oplus H_1, S, T, F)$
/* $k_1 = 0, f_0 = f$, and $k_0 = k$. */

1. Regarding s_1 and t_1 as *virtual* free vertices, find a $(k_0 - 1)$-DPC$[S_0 \setminus s_1, T_0 \setminus t_1|H_0, F_0]$.
2. If there exists a path P_i in the DPC which passes through both s_1 and t_1, let $P_i = (s_i, P_x, x, P_1, y, P_y, t_{\sigma_i})$, where P_1 is an s_1-t_1 path. If P_i and P_j pass through s_1 and t_1, respectively, let $P_i = (s_i, P_x, x, s_1, P_a, t_{\sigma_i})$ and $P_j = (s_j, P_b, t_1, y, P_y, t_{\sigma_j})$.
3. Find an $\bar{x}$-$\bar{y}$ hamiltonian path in H_1.
4. Merge the DPC and the hamiltonian path with edges $(x, \bar{x})$ and $(y, \bar{y})$.

Lemma 25. *When $k_1 = 0$, $f_0 = f$, and $k_0 = k$, Procedure DPC-E constructs an f-fault unpaired k-DPC for any $m \geq 6$ unless $k_0 = 2$ and $f_0 = m - 3$.*

Proof. It holds that $f_0 + (k_0 - 1) = f + k - 1 = m - 2$. If $k_0 - 1 \geq 2$, the $(k_0 - 1)$-DPC in Step 1 exists. If $k_0 - 1 = 1$, the $(k_0 - 1)$-DPC exists when $f_0 \leq m - 4$. Thus, the $(k_0 - 1)$-DPC exists unless $k_0 = 2$ and $f_0 = m - 3$. The existence of the $\bar{x}$-$\bar{y}$ hamiltonian path in Step 3 is straightforward. □

Lemma 26. *When $k_0 = 2$ and $f_0 = m - 3$, there exists an f-fault unpaired k-DPC in $H_0 \oplus H_1$ for any $m \geq 6$.*

Proof. There exists a hamiltonian cycle C_h in $H_0 \setminus F_0$. From C_h, we can construct four disjoint paths starting from the four terminals. If $C_h = (s_1, P_x, x, s_2, P_y, y, t_1, P_z, z, t_2, P_w, w)$, then it suffices to remove edges (x, s_2), (y, t_1), (z, t_2), and (w, s_1). The order of terminals in C_h does not matter. The four disjoint paths and 2-DPC$[\{\bar{x}, \bar{y}\}, \{\bar{z}, \bar{w}\}|H_1, \emptyset]$ are merged to obtain a desired 2-DPC. □

In the remaining part of Case 2, we assume $k_0 < k$. It implies $k_2 \geq 1$.

Procedure DPC-F$(H_0 \oplus H_1, S, T, F)$
/* $k_1 = 0, f_0 = f, k_0 < k$, and $k_0 \geq 1$ or $f_0 \geq 1$. */

1. Pick up $k_2 - 1$ free edges joining vertices in H_0 and vertices in H_1. Let X_0 be the set of endvertices of the free edges in H_0 and X_1 be in H_1.
2. Regarding s_1 as a virtual free vertex, find a $(k_0 + k_2 - 1)$-DPC$[S_0 \setminus s_1, T_0 \cup X_0|H_0, F_0]$. Assume path P_i in the DPC passes through s_1, and let $P_i = (s_i, P_a, x, s_1, P_b, t_{\sigma_i})$.
3. Case $\bar{x}$ is not a sink:
   (a) Find a k_2-DPC$[X_1 \cup \{\bar{x}\}, T_1|H_1, \emptyset]$.
   (b) Merge the two DPCs with the free edges and edge $(x, \bar{x})$.
4. Case $\bar{x}$ is a sink and $k_2 \geq 2$:
   (a) Find a $(k_2 - 1)$-DPC$[X_1, T_1 \setminus \bar{x}|H_1, \{\bar{x}\}]$.
   (b) Merge the two DPCs with the free edges and edge $(x, \bar{x})$.
5. Case $\bar{x}$ is a sink and $k_2 = 1$:
   (a) Pick up an edge (y, z) on a path in the DPC such that $y, z \neq x$.
   (b) Find a $\bar{y}$-$\bar{z}$ hamiltonian path in $H_1 \setminus \bar{x}$.
   (c) Merge the DPC and the hamiltonian path with edges $(x, \bar{x})$, $(y, \bar{y})$, and $(z, \bar{z})$.

Lemma 27. *When $k_1 = 0, f_0 = f, k_0 < k$, and $k_0 \geq 1$ or $f_0 \geq 1$, Procedure DPC-F constructs an f-fault unpaired k-DPC for any $m \geq 6$ unless (a) $k_0 = k_2 = 1$ and $f_0 = m - 3$, or (b) $k_2 = 2$ and $f_0 = m - 3$.*

Proof. The existence of k_2-1 free edges in Step 1 is straightforward. For Step 2, note that $f_0+(k_0+k_2-1) = f+k-1 = m-2$. If $k_0 + k_2 - 1 \geq 2$, the $(k_0 + k_2 - 1)$-DPC exists. Otherwise, it exists when $f_0 \leq m - 4$. Thus, the $(k_0 + k_2 - 1)$-DPC in Step 2 exists unless the exceptional cases (a) or (b). It holds that $k_2 < f_0 + k_0 + k_2 = f + k = m - 1$. Thus, the k_2-DPC in Step 3 exists if $k_2 \geq 2$. It also exists if $k_2 = 1$, due to Lemma 3. Similarly, we can see that the 1-fault $(k_2 - 1)$-DPC in Step 4 exists whether $k_2 - 1 \geq 2$ or not. The existence of the $\bar{y}$-$\bar{z}$ hamiltonian path in Step 5 is straightforward. Thus, we have the lemma. □

Lemma 28. *When $k_0 = k_2 = 1$ and $f_0 = m - 3$, there exists an f-fault unpaired k-DPC in $H_0 \oplus H_1$ for any $m \geq 6$.*

Proof. There exists a hamiltonian cycle $C_h = (s_1, P_x, x, s_2, P_y, y, t_1, P_z, z)$ in $H_0 \setminus F_0$. We decompose C_h into three disjoint paths starting from the three terminals in H_0: (s_1, P_x, x), (s_2, P_y, y), and (t_1, P_z, z). Let $\bar{z} \neq t_2$ first. If $\bar{x}, \bar{y} \neq t_2$, it suffices to find 2-DPC$[\{\bar{x}, \bar{y}\}, \{\bar{z}, t_2\}|H_1, \emptyset]$ and merge C_h and the 2-DPC. Otherwise, say $\bar{x} = t_2$, it suffices to find a $\bar{y}$-$\bar{z}$ hamiltonian path in $H_1 \setminus t_2$ and merge C_h and the hamiltonian path. Suppose $\bar{z} = t_2$. We use another representation of C_h, which is obtained by traversing C_h in reverse order. Let $C_h = (s_1, P_u, u, t_1, P_v, v, s_2, P_w, w)$. If $\bar{v} \neq t_2$, we can construct a desired DPC in the same way as before. Now, let $\bar{z} = \bar{v} = t_2$, which means $z = v = t_1$ and both (P_y, y) and (P_z, z) are empty. Then, $C_h = (s_1, P_x, x, s_2, t_1)$. It suffices to find an $\bar{x}$-t_2 hamiltonian path in H_1 and merge C_h and the hamiltonian path with edge $(x, \bar{x})$. The proof is completed. □

Lemma 29. *When $k_2 = 2$ and $f_0 = m - 3$, there exists an f-fault unpaired k-DPC in $H_0 \oplus H_1$ for any $m \geq 6$.*

Proof. There exists a hamiltonian cycle $C_h = (s_1, P_x, x, s_2, P_y, y)$ in $H_0 \setminus F_0$. We assume w.l.o.g. $\{\bar{x}, \bar{y}\} \neq \{t_1, t_2\}$. (Suppose otherwise, then we use another representation of C_h obtained by traversing C_h in reverse order.) If $\{\bar{x}, \bar{y}\} \cap \{t_1, t_2\} = \emptyset$, it suffices to find 2-DPC$[\{\bar{x}, \bar{y}\}, \{t_1, t_2\}|H_1, \emptyset]$ and merge C_h and the 2-DPC. If $|\{\bar{x}, \bar{y}\} \cap \{t_1, t_2\}| = 1$, say $\bar{x} = t_1$, it suffices to find a $\bar{y}$-t_2 hamiltonian path in $H_1 \setminus t_1$ and merge C_h and the hamiltonian path. This completes the proof. □

Case 3: $k_2 = k$ and $f = 0$.

In this case, all the sources are contained in H_0 and all the sinks are contained in H_1. There are no faults. By the assumption of $f + k = m - 1$, we have $k_2 = m - 1$. In the recursive structure of G^m, there are four components G_0, G_1, G_2, and G_3, which are $(m - 2)$-dimensional RC-like graphs. Unless all the $m - 1$ sources are contained in G_i and all the sinks are contained in $G_{(i+2) \bmod 4}$ for some i, letting H_0' (resp. H_1') be the subgraph induced by the vertices in G_1 and G_2 (resp. in G_3 and G_0), our problem is reduced to one of the two cases considered before. Thus, we assume w.l.o.g. that all the sources are contained in G_0 and all the sinks are contained in G_2.

The following procedure will construct an unpaired $(m - 1)$-DPC in which $m - 3$ paths pass through G_1 and do not pass through G_3. The remaining two paths in the DPC will pass through G_3. They may or may not pass through G_1.

Procedure DPC-G$([G_0 \oplus G_1] \oplus [G_2 \oplus G_3], S, T, F)$
/* $k = m - 1, f = 0, S \subset V(G_0)$, and $T \subset V(G_2)$. See Fig. 4. */

1. Let x be the vertex in G_0 such that $x^+ = {t_1}^-$.
2. Let $Z = \{t_1^+\} \cup \{u^+ : u \in V(G_2), (u, t_1) \in E(G_2)\}$. Pick up a vertex y in G_0 such that $y^- \notin Z$.
3. Pick up two sources, say s_{i_1} and s_{i_q} such that $\{s_{i_1}, s_{i_q}\} \cap \{x, y\} = \emptyset$. Regarding sources other than s_{i_1} and s_{i_q} as virtual free vertices, find a 2-DPC$[\{s_{i_1}, s_{i_q}\}, \{x, y\}|G_0, \emptyset]$.

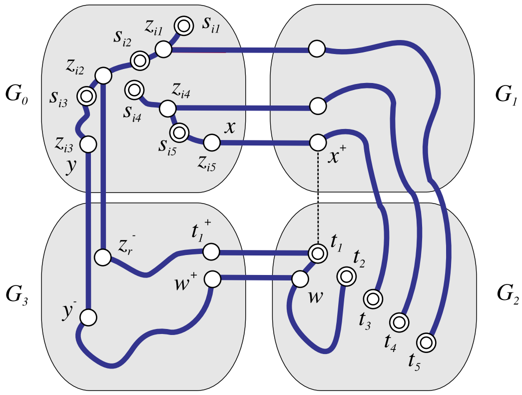

Fig. 4. Illustration of Procedure DPC-G.

4. Let s_{i_1}-path in the DPC be $(s_{i_1}, P_{i_1}, z_{i_1}, s_{i_2}, P_{i_2}, z_{i_2}, \ldots, s_{i_{q-1}}, P_{i_{q-1}}, z_{i_{q-1}})$, and let s_{i_q}-path be $(s_{i_q}, P_{i_q}, z_{i_q}, \ldots, s_{i_k}, P_{i_k}, z_{i_k})$, where $\{z_{i_{q-1}}, z_{i_k}\} = \{x, y\}$. Then, we have k disjoint s_j-z_j paths, $1 \le j \le k$, that cover $V(G_0)$.
5. Let r be an arbitrary index such that $r \ne i_{q-1}, i_k$. Let $Y = \{y^-, z_r^-\}$ and $X = \{z_j^+ : z_j \ne y, z_r\}$. Then, $Y \subset V(G_3)$ and $X \subset V(G_1)$.
6. Regarding t_1 as a virtual source, find an $(m-2)$-DPC$[X \cup \{t_1\}, T \setminus t_1 | G_1 \oplus G_2, \emptyset]$. Let the t_1-path in the DPC be (t_1, w, P_w, t_p) for some p.
7. Let $W = \{t_1^+, w^+\}$. If $z_r^- \notin W$, find a 2-DPC$[Y, W|G_3, \emptyset]$. If $z_r^- \in W$, find a 1-DPC$[\{y^-\}, W \setminus z_r^- | G_3, \{z_r^-\}]$.
8. Merge the three DPCs, the 2-DPC in G_0, the $(m-2)$-DPC in $G_1 \oplus G_2$, and the 2-DPC or 1-fault 1-DPC in G_3, with edges $\{(u^-, u) : u \in X\}$, $\{(u^+, u) : u \in Y\}$, and $\{(w, w^+), (t_1, t_1^+)\}$.

Lemma 30. *When $k = m - 1$, $f = 0$, $S \subset V(G_0)$, and $T \subset V(G_2)$, Procedure DPC-G constructs an f-fault unpaired k-DPC for any $m \ge 6$.*

Proof. The vertex y of Step 2 exists since $|Z| = m - 1 < 2^{m-2}$ for any $m \ge 6$. The degree $m - 2$ of G_0 is at least 4, thus the 2-DPC in G_0 exists by Lemma 19 and induction hypothesis. The s_j-z_j path in G_0 will be extended to pass through vertices in G_1 if $z_j^+ \in X$; otherwise, $z_j^- \in Y$ and the path will be extended to pass through vertices in G_3. Note that $|X| = m - 3$ and $|Y| = 2$. The existence of $(m-2)$-DPC in Step 6 is due to induction hypothesis. Recall that $G_1 \oplus G_2$ is an $(m-1)$-dimensional RC-like graph. By the choice of x in Step 1, t_1^- is a source of the $(m - 2)$-DPC in Step 6. Thus, w is certainly a vertex in G_2. Now, we have constructed $m - 3$ disjoint paths terminating at $T \setminus \{t_1, t_p\}$. To construct two paths terminating at $\{t_1, t_p\}$, Step 7 of the procedure works. Observe that $W \subset Z$ and $y^- \notin W$ by the choice of y. The 2-DPC in G_3 exists by Lemma 19 and induction hypothesis. The 1-fault 1-DPC in G_3 also exists by Lemma 3. This completes the proof. □

6. Concluding remarks

In this paper, it was shown that recursive circulant $G(2^m, 4)$ is f-fault one-to-one k-disjoint path coverable for any f and $k \ge 2$ with $f + k \le m$ when $m \ge 3$, and is f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \ge 2$ with $f + k \le m - 1$ when $m \ge 5$. The constructions presented in this paper are recursive and not so complicated. According to them, we can design efficient algorithms for finding the two types of disjoint path covers. Furthermore, the bound $f + k \le m$ for a one-to-one DPC problem and the bound $f + k \le m - 1$ for an unpaired DPC problem are both optimal.

It has been proven in [26] that $G(2^m, 4)$, $m \ge 4$, is f-fault *paired* many-to-many k-disjoint path coverable for any f and $k \ge 2$ with $f + 2k \le m$. For a graph G to be f-fault paired many-to-many k-disjoint path coverable, it is necessary that $\kappa(G) \ge f + 2k - 1$ [25]. The gap between the bound $f + 2k \le m$ for a paired DPC problem addressed in [26] and the bound $f + 2k \le m + 1$ of necessity is one. Recently, it was found that $G(32, 4)$ is 0-fault paired many-to-many 3-disjoint path coverable. It sheds light on the optimal construction of paired many-to-many disjoint path covers in $G(2^m, 4)$.

Appendix

Lemma 31. *$G(8, 4) \times K_2$ has a one-to-one 3-DPC for any $s, t \in V(H_0)$ when $f_1 = 1$ ($f_0 = f_2 = 0$).*

Proof. There exists a one-to-one 3-DPC in H_0 joining s and t, and there exists a hamiltonian cycle C_h in $H_1 \setminus F_1$. If there exists an edge (x, y) on some path in the DPC such that $(\bar{x}, \bar{y})$ is an edge of C_h, a desired one-to-one 3-DPC can be obtained by replacing (x, y) with $(x, \bar{x}, P_h, \bar{y}, y)$, where $P_h = C_h \setminus (\bar{x}, \bar{y})$. The number of edges in $G(8, 4)$ is 12. The 3-DPC passes through 9 edges and the hamiltonian cycle passes through at least 7 edges. Thus, at least four satisfy the required condition. □

Lemma 32. *$G(16, 4) \times K_2$ has a one-to-one 3-DPC for any $s, t \in V(H_0)$ when $F = \{\bar{s}, \bar{t}\}$.*

Proof. The proof is similar to that of Lemma 31. Recall the assumption of $(s, t) \notin E(G^m) \setminus F$. H_0 has a one-to-one 3-DPC, and $H_1 \setminus F_1$ has a hamiltonian cycle C_h. It suffices to show that there exists an edge (x, y) on some path in the DPC such that $(\bar{x}, \bar{y})$ is an edge of C_h. The number of edges in $G(16, 4)$ incident to neither s nor t is 24 (=32 − 4 · 2). The 3-DPC passes through 17 edges, among them 11 edges are incident to neither s nor t. The hamiltonian cycle passes through 14 edges, which are incident to neither $\bar{s}$ nor $\bar{t}$. Thus, there exists at least one edge satisfying the required condition. □

Lemma 33. *$G(8, 4) \times K_2$ has a one-to-one 3-DPC for any $s \in V(H_0)$ and $t \in V(H_1)$ when $F = \{\bar{t}\}$.*

Proof. By the assumption of $F \neq \{v_f\}$ for any v_f with $(s, v_f), (t, v_f) \in E(G^m)$, we have $(s, \bar{t}) \notin E(G^m)$. Let $V(H_0) = \{v_0, v_1, \ldots, v_7\}$ and $(v_i, v_j) \in E(H_0)$ if and only if $j \equiv i + 1$ or $i + 4 \pmod 8$. Assume w.l.o.g. $\bar{t} = v_0$ and $s \in \{v_2, v_3\}$. Since $H_0 \setminus F_0$ has a hamiltonian cycle $(v_1, v_2, v_6, v_7, v_3, v_4, v_5)$, we have a one-to-many 2-DPC in $H_0 \setminus F_0$ joining s and $\{v_1, v_5\}$. Furthermore, H_1 has a one-to-many 3-DPC $\mathcal{P}$ joining $\{\bar{s}, \bar{v_1}, \bar{v_5}\}$ and t as follows: for $s = v_2$, $\mathcal{P} = \{(\bar{s}, \bar{v_3}, \bar{v_4}, t), (\bar{v_1}, t), (\bar{v_5}, \bar{v_6}, \bar{v_7}, t)\}$; for $s = v_3$, $\mathcal{P} = \{(\bar{s}, \bar{v_2}, \bar{v_6}, \bar{v_7}, t), (\bar{v_1}, t), (\bar{v_5}, \bar{v_4}, t)\}$. A one-to-one 3-DPC in $H_0 \oplus H_1 \setminus F$ is obtained from the one-to-many 2-DPC in $H_0 \setminus F_0$ and the one-to-many 3-DPC in H_1. □

Lemma 34. *For any $s \in V(H_0)$ and $t \in V(H_1)$, $G(16, 4) \times K_2$ has a one-to-one 4-DPC when $F = \{\bar{t}\}$ and has a one-to-one 3-DPC when $F = \{\bar{t}, \bar{s}\}$.*

Proof. The proof is similar to that of Lemma 33. When $F = \{\bar{t}\}$, by the assumption of $F \neq \{v_f\}$ for any v_f with $(s, v_f), (t, v_f) \in E(G^m)$, we let $(s, \bar{t}) \notin E(G^m)$. When $F = \{\bar{t}, \bar{s}\}$, we also assume $(s, \bar{t}) \notin E(G^m)$; suppose otherwise, we can obtain a one-to-one 3-DPC from a 0-fault one-to-one 5-DPC in $G(16, 4) \times K_2$ without faulty elements by removing the two paths $(s, \bar{s}, t)$ and $(s, \bar{t}, t)$ from the 5-DPC. Notice that $(s, \bar{t})$ is an edge of $G(16, 4) \times K_2$ iff $(t, \bar{s})$ is an edge. Let $V(H_0) = \{v_0, v_1, \ldots, v_{15}\}$ and $(v_i, v_j) \in E(H_0)$ if and only if $j \equiv i + 1$ or $i + 4 \pmod{16}$. Assume w.l.o.g. $\bar{t} = v_0$ and $s \in \{v_2, v_3, v_5, v_6, v_7, v_8\}$. $H_0 \setminus F_0$ has a one-to-one 3-DPC between s and v_{15}. The vertices precede v_{15} on the three paths in the DPC are v_3, v_{11}, and v_{14}, which are the fault-free vertices adjacent to v_{15}. Therefore, there exists a one-to-many 3-DPC in $H_0 \setminus F_0$ joining s and $\{v_{11}, v_{14}, v_{15}\}$.

When $F = \{\bar{t}\}$, it suffices to construct a one-to-many 4-DPC $\mathcal{P}$ in H_1 joining $\{\bar{s}, \bar{v_{11}}, \bar{v_{14}}, \bar{v_{15}}\}$ and t as follows. Let $P_3 = (\bar{v_{14}}, \bar{v_{13}}, \bar{v_{12}}, t)$ and $P_4 = (\bar{v_{15}}, t)$.
For $s = v_2$, $\mathcal{P} = \{(\bar{s}, \bar{v_3}, \bar{v_4}, t), (\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_8}, \bar{v_7}, \bar{v_6}, \bar{v_5}, \bar{v_1}, t), P_3, P_4\}$;
for $s = v_3$, $\mathcal{P} = \{(\bar{s}, \bar{v_2}, \bar{v_1}, t), (\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_8}, \bar{v_7}, \bar{v_6}, \bar{v_5}, \bar{v_4}, t), P_3, P_4\}$;
for $s = v_5$, $\mathcal{P} = \{(\bar{s}, \bar{v_1}, t), (\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_8}, \bar{v_7}, \bar{v_6}, \bar{v_2}, \bar{v_3}, \bar{v_4}, t), P_3, P_4\}$;
for $s = v_6$, $\mathcal{P} = \{(\bar{s}, \bar{v_2}, \bar{v_3}, \bar{v_7}, \bar{v_8}, \bar{v_4}, t), (\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_5}, \bar{v_1}, t), P_3, P_4\}$;
for $s = v_7$, $\mathcal{P} = \{(\bar{s}, \bar{v_3}, \bar{v_2}, \bar{v_6}, \bar{v_5}, \bar{v_1}, t), (\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_8}, \bar{v_4}, t), P_3, P_4\}$;
for $s = v_8$, $\mathcal{P} = \{(\bar{s}, \bar{v_7}, \bar{v_3}, \bar{v_4}, t), (\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_5}, \bar{v_6}, \bar{v_2}, \bar{v_1}, t), P_3, P_4\}$.

When $F = \{\bar{t}, \bar{s}\}$, it suffices to construct a one-to-many 3-DPC $\mathcal{P}'$ in $H_1 \setminus \bar{s}$ joining $\{\bar{v_{11}}, \bar{v_{14}}, \bar{v_{15}}\}$ and t as follows.
For $s = v_2$, $\mathcal{P}' = \{(\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_8}, \bar{v_4}, \bar{v_3}, \bar{v_7}, \bar{v_6}, \bar{v_5}, \bar{v_1}, t), P_3, P_4\}$;
for $s = v_3$, $\mathcal{P}' = \{(\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_8}, \bar{v_7}, \bar{v_6}, \bar{v_2}, \bar{v_1}, \bar{v_5}, \bar{v_4}, t), P_3, P_4\}$;
for $s = v_5$, $\mathcal{P}' = \{(\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_8}, \bar{v_4}, \bar{v_3}, \bar{v_7}, \bar{v_6}, \bar{v_2}, \bar{v_1}, t), P_3, P_4\}$;
for $s = v_6$, $\mathcal{P}' = \{(\bar{v_{11}}, \bar{v_{10}}, \bar{v_9}, \bar{v_5}, \bar{v_4}, \bar{v_8}, \bar{v_7}, \bar{v_3}, \bar{v_2}, \bar{v_1}, t), P_3, P_4\}$;
for $s = v_7$, $\mathcal{P}' = \{(\bar{v_{11}}, \bar{v_{10}}, \bar{v_6}, \bar{v_5}, \bar{v_9}, \bar{v_8}, \bar{v_4}, \bar{v_3}, \bar{v_2}, \bar{v_1}, t), P_3, P_4\}$;
for $s = v_8$, $\mathcal{P}' = \{(\bar{v_{11}}, \bar{v_7}, \bar{v_6}, \bar{v_{10}}, \bar{v_9}, \bar{v_5}, \bar{v_4}, \bar{v_3}, \bar{v_2}, \bar{v_1}, t), P_3, P_4\}$. □

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